

# Atick-Witten Hagedorn Conjecture, near scale-invariant matter and blue-tilted gravity power spectrum

Tirthabir Biswas<sup>1</sup>, Tomi Koivisto<sup>2</sup>, and Anupam Mazumdar<sup>3</sup>

<sup>1</sup>*Department of Physics, Loyola University, New Orleans, LA 70118*

<sup>2</sup>*Nordita, KTH Royal Institute of Technology and Stockholm University, Roslagstullsbacken 23, SE-10691 Stockholm, Sweden*

<sup>3</sup>*Consortium for Fundamental Physics, Physics Department, Lancaster University, LA1 4YB, UK*

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We will provide an interesting new mechanism to generate almost scale invariant seed density perturbations with a *red spectrum*, while keeping the gravitational wave spectrum *blue-tilted* in a stringy thermal contracting phase at temperatures beyond the Hagedorn temperature. This phase is often referred to as the Hagedorn phase where the free energy has been conjectured by Atick and Witten to grow more slowly than ordinary radiation. The primordial fluctuations are created by the statistical thermal fluctuations determined by the partition function, rather than quantum vacuum driven fluid dynamical fluctuations. In order for our mechanism to work we require a non-singular bouncing cosmology.

There has been a very important discovery of B-mode of polarisation from the BICEP team [1], which points its origin towards the primordial gravitational waves seeded during the earliest epochs in the history of the universe, roughly at the scale close to the  $10^{16}$  GeV. The data suggests a tensor-to-scalar ratio  $0.15 \leq r(k_*) \equiv \mathcal{P}_\zeta(k_*)/\mathcal{P}_h(k_*) \leq 0.27$ , at the pivot scale,  $k_* = 0.002 \text{ Mpc}^{-1}$ . There is a mild hint that the gravitational wave spectrum is growing with a *blue-tilted* spectrum, while Planck [2] and WMAP [3] has provided evidence for a red tilted matter power spectrum. Typically, in inflationary models it is challenging to generate a *blue tilted* gravitational wave. The amplitude of the gravitational waves is proportional to the total energy density of the universe, the energy density needs to grow as progressively shorter modes leave the Hubble patch during inflation. However, this is impossible in a monotonically expanding universe as long as the matter content satisfies the weak energy condition. For instance, the Hubble expansion rate decreases very gradually during typical inflation, for a review, see [4]. Now, in bouncing (cyclic) cosmologies there is (are) *contracting phase(s)* preceding *expanding phase(s)* where the energy density increases. This therefore naturally gives rise to a *blue-tilted* gravitational wave spectrum. The challenging part, actually, is to produce a near-scale-invariant red-tilted scalar spectrum. In this letter we will show that a contracting Hagedorn phase is precisely what is needed to generate a matter power spectrum that remains *red-tilted*, while the gravity spectrum grows with increasing multipoles. Moreover, the proposed model can give rise to a relatively large tensor-to-scalar ratio.

One of the most intriguing features of string theory is the existence of the Hagedorn phase at high temperatures where the energy is not dominated by the massless modes, but rather by the most *massive string states*, leading to a pressureless fluid [5–7]. In fact, a canonical description of the thermal phase indicated a limiting Hagedorn temperature [5]. Later, however, it was argued that the limiting temperature only corresponds to

the emergence of a thermal tachyonic mode making the description of the system in terms of fundamental string excitations invalid [8]. It was further conjectured by Atick and Witten in one of the classic papers [9] that at temperatures larger than the Hagedorn temperature, the free energy  $\mathcal{F}$  grows much more slowly,

$$\mathcal{F} \propto T^2, \quad (1)$$

as compared to conventional field theories where  $\mathcal{F} \propto T^4$ . Therefore, the system represents many fewer degrees of freedom than one would have expected from the zero-temperature string spectrum, or even in point-like particle field theories. It is worth pointing out that a finite temperature loop calculation [10] of a toy  $p$ -adic string model precisely exhibited the  $T^2$  behavior along with thermal duality:  $Z(T) \sim Z(T_H^2/T)$ , another feature also conjectured in [9].  $Z$  denotes the finite temperature partition function and  $T_H$ , the Hagedorn temperature, is related to the string scale via  $\mathcal{O}(1)$  factors. Finally, such a “stiff fluid” (equation of state parameter,  $w = p/\rho \approx 1$ ) behaviour (1) has also been argued to emerge in a quantum gravity phase of interconnected blackholes [11]. What is rather intriguing is that (1) seems to lead to a new mechanism to generate near scale-invariant matter perturbations.

We should point out that our scenario is completely different from the Brandenberger-Nayeri-Vafa mechanism [12], which is based on the behaviour of closed string modes below the Hagedorn temperature. In our case we are looking at stringy thermodynamics above the Hagedorn temperature. Moreover, the mechanism in [12] requires a loitering or a slow bounce phase [13] to realize, which one has to invoke new physics beyond Einstein’s theory of General Relativity (GR), whereas the mechanism we will discuss is based on a contracting universe dominated by a stiff fluid and evolving according to GR. We do of course, require a bounce mechanism (see e.g. [14] and [15]) to transition from contraction to expansion, but the modes that we are observing today at CMB exits prior to the bounce. Our scenario is also very distinct

from the ekpyrotic case [16] which operates at  $w \gg 1$  whereas we have  $w \approx 1$ . The origin of the perturbations is also very different, we are considering thermal fluctuations as opposed to scalar field fluctuations.

**Thermal Fluctuations:** The possibility of thermal fluctuations as the origin of small inhomogeneities and anisotropies in the cosmic microwave background perhaps dates back to Peebles [17]. In general, fluid fluctuations can arise from two different sources:

- There could be fluctuations in the energy density and the associated temperature. This can arise, for instance, due to quantum vacuum fluctuations, and this is what has been traditionally discussed in the literature, see [18].
- However, even if one can define a unique temperature in a given volume, there are fluctuations in energy within this volume due to the very statistical nature of thermal physics. This could also potentially seed primordial fluctuations, see for instance [19] and references there-in.

The statistical fluctuations in the energy inside a given volume  $L^3$  is given by

$$\begin{aligned} \langle \Delta E \rangle_L^2 &\equiv \langle E^2 \rangle - \langle E \rangle^2 = \frac{\partial^2 \ln Z}{\partial \beta^2} = T^2 C_L \\ \Rightarrow \langle \delta \rho^2 \rangle_L &= \frac{T^2 C_V}{L^6} = \frac{T^2}{L^3} \frac{\partial \rho}{\partial T} \end{aligned} \quad (2)$$

where  $C_L$  is the heat capacity of the thermal system for a given volume  $L^3$ . These are random fluctuations that exists in any finite temperature system and arise already at the classical level as long as the fluid is in local thermal equilibrium. The power spectrum for the seed perturbations will then be primarily given by these thermal fluctuations till the wavelengths of the fluctuations are smaller than the Hubble length. Once the modes become super-Hubble, thermal correlations over the relevant physical wavelengths can no longer be maintained, instead the fluctuations evolve according to the usual hydrodynamical differential equations, coupling the metric and matter fluctuations. Essentially, this is a set-up where the sub-Hubble thermal fluctuations (instead of the traditional quantum vacuum fluctuations) act as the initial conditions to seed the super-Hubble fluctuations.

Very recently in [20] a precise understanding of how these statistical fluctuations get encoded in the curvature perturbations,  $\zeta$ , at the ‘‘Hubble crossing’’, and explicit formulae were obtained for a general extensive thermodynamic fluid whose pressure,  $p$ , can be an arbitrary function of the temperature,  $T$ . The curvature power spectrum is given by

$$\mathcal{P}_\zeta = k^3 \zeta_k^2 = \sqrt{3} \gamma^2 A^2(T_k) \frac{T_k^2 \rho'_k}{M_p^3 \sqrt{\rho_k}}, \quad (3)$$

where  $\gamma = 2\sqrt{2}\pi^{3/4} \approx 6.7$  and  $M_p$  is the reduced Planck mass. The subscript  $k$  (which we are going to subsequently drop) refers to the fact that all these quantities

have to be evaluated at the Hubble crossing condition,  $H_k = k/a$ . We are using the notations described in [20] where

$$A(T) = \frac{3(1+w)\Omega + 2(3+r)}{6(1+w)\Omega}, \quad (4)$$

and

$$r = -\frac{3}{2} \left[ 1 + \frac{(1+w)\rho(2\rho' + T\rho'')}{T\rho'^2} \right]. \quad (5)$$

Note that all the above functions of temperature can be calculated if we know  $p(T)$  as the energy density is related rather straightforwardly to pressure:

$$\rho(T) = T \frac{dp(T)}{dT} - p(T) \quad (6)$$

Just to illustrate, for radiation the above formula yields

$$\mathcal{P}_\zeta = \frac{\sqrt{3}g\gamma^2}{4} \left( \frac{T}{M_p} \right)^3, \quad (7)$$

where  $g$  is the number of degrees of massless modes, and  $T$  corresponds to the temperature when the given mode becomes super-Hubble. Evidently, the spectrum depends strongly on the temperature, and  $T \propto 1/a$  gives rise to very large blue tilt.

**Hagedorn Phase and CMB Spectrum:** According to the Atick-Witten conjecture, the partition function only grows as  $T^2$  at high temperatures, it is then natural to assume the pressure to be of the form

$$p(T) = M_s^4 \left[ \left( \frac{T}{M_s} \right)^2 + c_1 \left( \frac{T}{M_s} \right) + c_2 \ln \left( \frac{T}{M_s} \right) \right], \quad (8)$$

with subleading linear and log terms. Note that Atick-Witten behaviour can be expected to hold above the Hagedorn temperature which should be close to  $M_s$ , both being expected to be related to the string scale by numerical factors [9, 10]. It is a little more transparent and convenient to work with a slightly different functional form

$$p(T) = M_s^4 \left[ \left( \frac{T}{M_s} \right)^2 + c_1 \left( \frac{T}{M_s} \right)^{1+\alpha} \right] \text{ with } |\alpha| \ll 1. \quad (9)$$

Hence, here we will focus on Eq. (9) while the analysis with Eq. (8) and more general subleading corrections will be provided elsewhere.

The spectrum can be calculated very straightforwardly, and one obtains:

$$\begin{aligned} \mathcal{P}_\zeta &= \frac{\sqrt{3}\gamma^2}{32} (1 + \alpha - 3\alpha^2 + \alpha^3)^2 c_1^2 \left( \frac{M_s}{M_p} \right)^3 \left( \frac{T}{M_s} \right)^{2\alpha} \\ &+ \mathcal{O} \left( \frac{T}{M_s} \right)^{-1+\alpha} \approx \frac{\sqrt{3}\pi^{\frac{3}{2}} c_1^2}{4} \left( \frac{M_s}{M_p} \right)^3 \left( \frac{T}{M_s} \right)^{2\alpha} \end{aligned} \quad (10)$$

As  $\alpha \rightarrow 0$  we have a scale-invariant spectrum. If  $\alpha > 0$  the spectrum has a blue tilt, but if  $\alpha < 0$ , the spectrum will be red-tilted. Since for a stiff fluid,  $\rho \sim H^2 \sim T^2 \sim a^{-6}$ , the spectral tilt can be calculated quite easily:

$$1 - n_s = -3\alpha \quad (11)$$

If  $c_1 \sim \mathcal{O}(1)$ , to reproduce the correct amplitude of the power spectrum we should have  $M_s/M_p \sim \mathcal{O}(10^{-3} - 10^{-4})$ . These numbers are very reasonable and as one can see there is no fine-tuning required to generate the near scale invariance of the spectrum. One only requires the subleading correction to be close to being linear, a very natural assumption. In our opinion, this is a real advantage of our mechanism over standard inflationary models as it bypasses the need to have a very flat potential with small slow-roll parameters!

**Gravitational Waves:** The initial conditions for the primordial gravitational waves are set by quantum vacuum fluctuations since extensive thermal matter does not provide any additional source for gravitational waves. The gravitational wave spectrum is thus given by [20]

$$\mathcal{P}_h = \frac{1}{4\pi^2} \left( \frac{H}{M_p} \right)^2 = \frac{\rho}{12\pi^2 M_p^4}. \quad (12)$$

The tensor to scalar ratio becomes

$$r_{t/s} \equiv \frac{\mathcal{P}_h}{\mathcal{P}_\zeta} \approx \frac{1}{3\pi^3 \sqrt{3}\pi c_1^2} \left( \frac{M_s}{M_p} \right) \left( \frac{T}{M_s} \right)^{2(1-\alpha)}. \quad (13)$$

It is clear that there is a large parameter space where  $r$  is going to be appreciable. For instance, if  $c_1 \sim 0.1$ , then in the region  $T \sim \sqrt{M_s M_p}$  we will get  $r \sim \mathcal{O}(1)$ .

Let us now consider the predictions in some details. To match with the observed CMB amplitude [2],  $A_0 \approx 2.4 \cdot 10^{-9}$ , we fix the parameter  $c_1$  as

$$c_1 = \frac{2\sqrt{A_0}}{3^{\frac{1}{4}}\pi^{\frac{3}{4}}} \left( \frac{M_s}{M_p} \right)^{-\frac{3}{2}} \left( \frac{T_{pivot}}{M_p} \right)^{-\alpha}. \quad (14)$$

In the following, we will use  $T_{pivot} = 0.01 M_p$ . In Fig. 1 we show the scalar spectrum and the  $r$  as functions of temperature when  $M_s = 10^{-4} M_p$ <sup>1</sup>. The value of  $c_1$  is adjusted to match with the CMB power-spectrum according to (14), and we have used  $\alpha = -4/300$  to obtain the central value of the spectral index,  $n_s = 0.96$ , as observed by the PLANCK collaboration [2]. The interesting regime is  $T \gtrsim M_s$ , much below this, the spectrum

is not scale-invariant, gravitational waves are unobservably small, and in any case we expect (9) to be valid only when  $T \gtrsim M_s$ . In the Fig. 2, we have provided numerical plots of  $r$  as a function of the ratios  $T/M_p$  and  $M_s/M_p$ . The value of  $c_1$  is adjusted to match with the CMB power-spectrum. As expected, we find large parameter spaces where  $r$  can be considerable.

**Spectral tilts & Low Multipole anomalies:** It is clear from the plots that the gravity waves have a strong blue tilt in our model which discriminates it from inflationary ones. In fact, it is easy to calculate the spectral tilt: Since  $\mathcal{P}_h \sim T^2$

$$n_t = d \ln \mathcal{P}_h / d \ln k \approx 3 \quad (15)$$

which is consistent with the value,  $n_t = 1.5261^{+3.4739}_{-3.5261}$ , that was quoted in a recent analysis [21] performed by combining the BICEP2 data with Planck, WMAP, and BAO. We would like to point out however, an important twist that a Planck scale bounce, that is anyway required for the success of the mechanism, may provide. While in the GR contracting phase  $|H|$  increases, as the temperatures approach the Planckian regime, the increase must taper down (and eventually  $H$  should start to decrease to reach  $H = 0$  at the bounce point). Since  $\mathcal{P}_h \propto H^2$ , this would essentially decrease the observed  $|n_t|$  (or give rise to a negative running of the tilt), the details of which will depend on the nature of the bouncing background cosmology and requires further study. We note in passing that no assumption about the validity of GR is made while deriving the thermal power-spectrum (3) and it should therefore be also valid near the bounce.

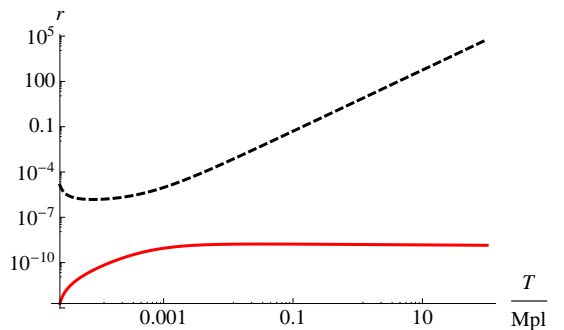


FIG. 1: The scalar spectrum  $\mathcal{P}_\zeta$  (black dashed line) and  $r$  as (red solid line) as functions of the temperature  $T/M_p$ . Here  $M = 10^{-4} M_p$ , and  $c_1 \approx 30$  and  $\alpha = -0.075$ .

Finally, there is one last interesting possibility worth mentioning. Once  $T \lesssim T_H$ , we do not expect (9) to remain valid, but rather to see a sharper increase of pressure with temperature. This would change the dependence of the matter spectrum on temperature. In fact, for any  $p \propto T^{2+\beta}$  with  $\beta > 0$ , one obtains a blue spectrum. In other words, at around  $T \sim M_s \sim T_H$ , we expect to see a transition of the matter spectrum from a blue to a red tilt! It is tempting to investigate whether

<sup>1</sup> The observed scales in CMB sky approximately span three orders of magnitude. Minimally we want all these modes to exit during the Hagedorn contraction phase, and since  $T_k \propto k^{3/2}$ , this corresponds to a temperature range spanning approximately  $\mathcal{O}(10^4 - 10^5)$ . Since we expect (9) to remain valid within  $T_H \sim M_s \lesssim T \lesssim \mathcal{O}(10) M_p \sim \text{Planck mass}$ , there is a separate good reason to choose  $M_s/M_p \propto \mathcal{O}(10^{-3} - 10^{-4})$ .

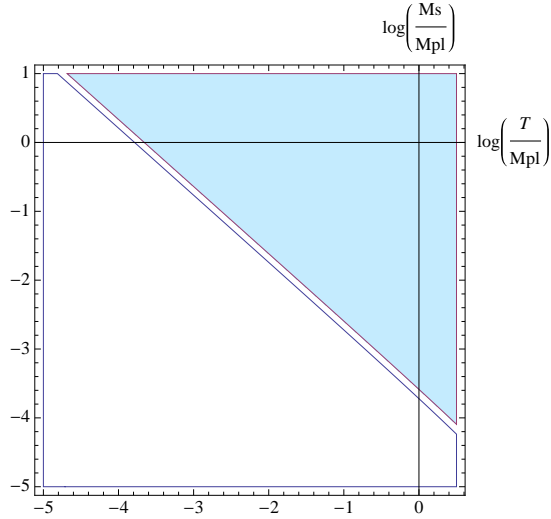


FIG. 2: The region of the parameter space where  $0.15 < r < 0.27$  corresponding constraints from BICEP2 [1]. In the blue region we have  $r > 0.27$ . The observable CMB scales span about 4 orders of magnitude on the vertical temperature axis for a given  $M_s$ .

this feature can provide an explanation for the low multipole anomalies, the fact that the matter spectrum seems to rise between  $\ell = 2$  to  $\ell \sim 40$  [2].

**Closing Remarks:** Last, but not the least, it is important to highlight that these statistical fluctuations in the Hagedorn phase need to be stretched to seed the initial perturbations for the structure formation. This could be arranged if the universe underwent a phase of inflationary expansion afterwards (need not be near exponen-

tial, a power-law growth may suffice) or several phases of asymmetric cyclic growth as in *cyclic-inflation* [22]. For the cyclic inflation case, if the entropy production is large, then potentially all the CMB modes approximately spanning 3 orders of magnitude ( $\ell = 2$  to  $\ell \sim 2500$ ) can exit within the same cycle, and all our previous analysis then remains valid. If on the other hand, the entropy production is small, then essentially the modes will exit approximately at the same temperature but in subsequent cycles. In this case, the spectral tilts will depend on how the turnaround energy scale changes in subsequent cycles. This is typically governed by underlying scalar field evolution [22] and the spectral-tilt calculations will need to be revisited.

To summarize, we have considered a very simple thermal universe motivated by stringy physics. It provides a near scale invariant matter spectrum which could be red-tilted, and predicts a blue tilted gravity wave spectrum. Additionally, the expected range of the Hagedorn phase (in orders of magnitude) is tantalizingly close to what we are currently able to access in the CMB sky leading to the possibility of being able to see the “edge effects”: a power enhancement at low multipoles as the Hagedorn phase gives way to more traditional growth of pressure with temperature, and a running of the gravitational spectral tilt at high multipoles as imprints of a quantum gravitational bounce!

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